Proposed Control Approach Quasi-Sliding Mode Control

Lodhi Sadam Sadiq Khan[#1], Du Bao Zhu[*2], Naveed Ahmed Qureshi [#3]Ibrajimov Shokhzod[#4]

[#]School Of Automation, Nanjing University of Science & Technology, Nanjing PR. China.

***Abstract* —** *Time delays and external disturbances are unavoidable in many practical control applications, e.g., in robotics, manufacturing, and process control and it is often a source of instability or oscillations, see, e.g., [1,2] and the references therein. Therefore, the design of control and observation schemes has been an interesting problem for dynamical systems to compensate for time delays [3] and to estimate external disturbances [4]. To enhance robustness, the sliding mode control methodology has been recognised as an effective strategy for uncertain systems, see, e.g., and references therein. In this context, there have been considerable efforts devoted to the problem of sliding mode control design for uncertain systems with matched disturbances, see, e.g., [5,6] and references therein. However, when the matching conditions for disturbances are not satisfied, their effects can be only partially rejected in the sliding mode. Therefore, the control design for this case remains a challenging problem.*

For a class of linear systems with time-varying delay and unmatched disturbances, a sliding-mode control strategy was developed in and sufficient conditions were derived in terms of linear matrix inequalities (LMIs) to guarantee that the state trajectories of the system converge towards a ball with a pre-specified convergence rate. By using the invariant ellipsoid method, another sliding mode control design algorithm was proposed for a class of linear quasi-Lipschitz disturbed system to minimise the effects of unmatched disturbances to system motions in the sliding mode . Later, by combining the predictor-based sliding mode control with the invariant ellipsoid method, an improved result was reported to take into account also time delay in the control input [10]. Recently, a disturbance observer-based sliding mode control was presented in where mismatched uncertainties were considered.

Keywords — *Quasi-Sliding, Model Control, Time-Delay Systems, Lyapunov Functionals.*

I. INTRODUCTION

Owing to advantages of digital technology, there has been increasing attention paid to the discrete-time sliding mode control. In [12], the quasi-sliding mode control and the associated quasi-sliding mode band (QSMB) and reaching law were introduced for single input discrete systems. Another quasi-sliding mode control design algorithm was reported in [11], adopting a different reaching law. A discrete-time sliding mode controller was synthesised to drive the system state trajectories into a small bounded region for a class of linear multi-input systems with matching perturbations [13]. A robust quasi-sliding mode control strategy was proposed in [61] for uncertain systems using multirate output feedback. In , a predictor-based sliding mode control law was used to deal with discrete-time uncertain systems subject also to an input delay. In , a sufficient condition for the existence of stable sliding surfaces, depending on the lower and upper delay bounds, was derived in terms of LMIs. Recently, some improved results for this problem have been reported in [7,8,9].

In the framework of discrete-time sliding mode control, the problem of compensation for time-varying delay and rejection of the unmatched disturbance effects has not received much attention and so it will be addressed in this chapter. Here, by using the L-K method, in combination with the reciprocally convex approach, sufficient conditions for the existence of a stable sliding surface are derived in terms of LMIs. Moreover, these conditions guarantee that the effects of interval time-varying delay and unmatched disturbances are mitigated, and the induced sliding dynamics are exponentially convergent within a ball with a radius to be minimised. A robust discrete-time quasi-sliding mode is then synthesised to drive the state trajectories of the closed-loop system towards the prescribed sliding surface and remain in this ball after a finite time.

The paper is organized as follows. After the introduction, presents the system definition and some preliminaries. The main results are included. The effectiveness of the proposed control approach is illustrated through numerical examples. Finally, concludes of the paper

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider a class of linear discrete-time uncertain systems described in the following form

$$x(k + 1) = Ax(k) + Adx(k - \tau(k)) + Bu(k) + D\omega(k),$$

$$x(k) \equiv \varphi(k), \qquad k \in Z[-\tau M,0],$$

where x(k) ∈ Rn and u(k) ∈ Rm are, respectively, the system state vector and the control input. Matrices A,Ad,B and D are constant, with appropriate dimensions, where rank(B) = m ≤ n. The initial function of system, φ(k),k ∈ Z[−τM,0], has its norm given by

$$\|\phi\|_{\tau} = \max\{\|\phi(k)\| : k \in \mathbb{Z}[-\tau_M, 0]\}.$$

The delay τ(k) is time-varying delay in the whole process and satisfying

0 ≤ τm ≤ τ(k) ≤ τM,

where τm and τM satisfying τm < τM, are known positive integers representing, respectively, the minimum and maximum delay bounds. The unmatched external disturbance ω(k) ∈ Rp is assumed to be bounded, i.e., for any k ≥ 0,

$$\omega^T(k)\omega(k) \leq \omega_p^2, \quad \forall k \geq 0,$$

where ωp is a positive scalar.
It can be shown that if B is a full-column rank matrix, i.e., rank(B) = m, there exists a non-singular transformation matrix T which can always be chosen such that

$$TB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix},$$

where B2 ∈ Rm×m is a non-singular matrix [71]. With z(k) = Tx(k), system can be transformed into the following regular form:

$$z(k+1) = \overline{A}z(k) + \overline{A_d}z(k-\tau(k)) + \overline{B}u(k) + \overline{D}\omega(k)$$

where

$$\overline{A} = TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \overline{A}_d = TA_dT^{-1} \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix}$$

$$\overline{B} = TB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \overline{D} = TD = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}.$$

,

Now, by partitioning z(k) = [z1(k) z2(k)]T , where z1(k) ∈ Rn−m and z2(k) ∈ Rm, the dynamics of system

can be described by

$$z_1(k+1) = \sum_{i=1}^{2}(A_{1i}z_i(k) + A_{d1i}z_i(t-\tau(k))) + D_1\omega(k),$$
$$z_2(k+1) = \sum_{i=1}^{2}(A_{2i}z_i(k) + A_{d2i}z_i(t-\tau(k))) + B_2u(k) + D_2\omega(k))$$

The main purpose is first to derive sufficient conditions for the existence of a stable sliding surface such that in the induced sliding dynamics, the effects of time-varying delay and unmatched disturbances can be mitigated. These conditions also guarantee that all the state trajectories are exponentially convergent to a ball whose radius can be minimised. Finally, a discrete-time quasi-sliding mode controller is proposed to drive the system state trajectories to the quasi-sliding mode.

III. ROBUST QUASI-SLIDING MODE CONTROL DESIGN

A. Sliding function design

The sliding function for system is proposed as follows,

s(k) = Cz(k) = [−C I]z(k) = −Cz1(k) + z2(k), (5.7)

where C ∈ Rm×(n−m) is a constant matrix to be designed. In the induced sliding mode, we have s(k) = 0 so that z2(k) = Cz1(k). The reduced-order sliding motion can thus be obtained as

z1(k + 1) = [A11 + A12C]z1(k) + [Ad11 + Ad12C]z1(k − τ(k)) + D1ω(k).

Note that the sliding surface design is now equivalent to the stabilisation problem for system (A11,A12,Ad11,Ad12) where (A11,A12) and (Ad11,Ad12) are assumed to be controllable. Reduced-order system will be stabilised by choosing an appropriate matrix C. Due to the presence of the unmatched disturbances ω(k), in general, the asymptotic convergence of state trajectories of system cannot be achieved. In that case, instead of investigating asymptotic stability of the system, we consider the system state convergence within the neighborhood of the equilibrium point. However, the shape of such a neighborhood is, in general, very complex and hard to determine exactly. Hence, the estimation of outer or inner bounding simple convex shapes as balls or ellipsoids or boxes will be considered. This is formalised in term of the existence problem, which must be solved to determine the switching surface.

In the following, for the sake of simplicity, we denote e1 = [In−m 0(n−m)×8(n−m)], ei = [0(n−m)×(i−1)(n−m) In−m 0(n−m)×(9−i)(n−m)], i = 2,3,...,8, e9 = [0(n−m)×8(n−m) In−m] as entry matrices. The following notations are specifically used in our development. For given integers τm,τM satisfying 0 < τm < τM, any scalar λ, nonsingular matrix K ∈ R(n−m)×(n−m),F = K−1, matrices X,G, and symmetric positive definite matrices P,Qj,Rj,Sj,j = 1,2 of appropriate dimensions, we denote the following vectors.

$$y(k) = z_1(k+1) - z_1(k), \rho(k) = \left[z_1^T(k)F^T \ y^T(k)F^T\right]^T$$

$$\xi(k) = \Big[z_1^T(k)F^T \ z_1^T(k-\tau_m)F^T \ z_1^T(k-\tau(k))F^T \ z_1^T(k-\tau_M)F^T \sum_{s=k-\tau_m}^{k-1} z_1^T(s)F^T \sum_{s=k-\tau(k)}^{k-\tau_m-1} z_1^T(s)F^T \sum_{s=k-\tau_M}^{k-\tau(k)-1} z_1^T(s)F^T \ y^T(k)F^T \ \omega^T(k)\Big]^T,$$

$$\zeta(k) = \left[z_1^T(k)F^T \sum_{s=k-\tau_m}^{k-1} z_1^T(s)F^T \sum_{s=k-\tau_M}^{k-\tau_m-1} z_1^T(s)F^T\right]^T,$$

Constants

$$\tau_a = \frac{\tau_m(\tau_m+1)}{2}, \tau_b = \frac{(\tau_M-\tau_m)(\tau_M+\tau_m+1)}{2}$$

$$\alpha_1 = \frac{1-\alpha}{\alpha-\alpha^{\tau_m+1}}, \alpha_2 = \frac{1-\alpha}{\alpha^{\tau_m+1}-\alpha^{\tau_M+1}},$$

$$\alpha_3 = \frac{(1-\alpha)^2}{\alpha\left[\alpha^{1+\tau_m}-(1+\tau_m)\alpha+\tau_m\right]},$$

$$\alpha_4 = \frac{(1-\alpha)^2}{\alpha\left[\alpha^{1+\tau_M}-\alpha^{1+\tau_m}+(1-\alpha)(\tau_M-\tau_m)\right]},$$

And Matrices

$$G = CK, \Gamma = e_1^T + \lambda e_8^T, R_c = \tau_m R_1 + (\tau_M - \tau_m)R_2,$$

$$\mathcal{A}_c = [A_{11}K + A_{12}G - K \ \ 0_{n-m} \ \ A_{d11}K + A_{d12}G \ \ 0_{(n-m)\times 4(n-m)} \ \ -K \ \ D_1]$$

$$\Pi_1 = e_1 + e_8, \Pi_2 = \left[e_1^T \ \ e_8^T\right]^T, \Pi_3 = \left[e_5^T \ \ (e_1-e_2)^T\right]^T,$$

$$\Pi_4 = \left[e_6^T \ \ (e_2-e_3)^T \ \ e_7^T \ \ (e_3-e_4)^T\right]^T,$$

$$\Pi_5 = \tau_m e_1 - e_5, \Pi_6 = (\tau_M - \tau_m)e_1 - e_6 - e_7.$$

Now, we are ready to present the first theorem that gives sufficient conditions for the existence of a stable sliding surface as follows.

Theorem For system with given positive integers τm and τM for the delay, where 0 < τm < τM, and disturbance bound ωp > 0, if there exist scalars λ and α, where α > 1, a nonsingular matrix K = F −1, matrices X,G and symmetric positive-definite matrices P,Qj,Rj,Sj,j = 1,2, of appropriate dimensions such that the following inequalities hold

$$\Omega(\alpha) < 0, \qquad)$$

$$\begin{bmatrix} R_2 & X \\ \star & R_2 \end{bmatrix} \geq 0 \qquad ,$$

where

$$\Omega(\alpha) = \Pi_1^T P \Pi_1 - \alpha^{-1} e_1^T P e_1 + e_1^T Q_1 e_1 + \alpha^{-\tau_m} e_2^T (Q_2 - Q_1) e_2 - \alpha^{-\tau_M} e_4^T Q_2 e_4 + \Pi_2^T R_c \Pi_2 - \alpha_1 \Pi_3^T R_1 \Pi_3 - \alpha_2 \Pi_4^T \begin{bmatrix} R_2 & X \\ \star & R_2 \end{bmatrix} \Pi_4 + e_8^T(\tau_a S_1 + \tau_b S_2) e_8 - \alpha_3 \Pi_5^T S_1 \Pi_5 - \alpha_4 \Pi_6^T S_2 \Pi_6 + \Gamma \mathcal{A}_c + \mathcal{A}_c^T \Gamma^T - (1-\alpha^{-1}) e_9^T e_9,$$

then the state trajectories of the sliding dynamics are exponentially convergent within a ball B(0,r) with radius

$$r = \frac{\overline{\omega}_p}{\sqrt{\lambda_{min}(F^T P F)}}.$$

Moreover, the design matrix C can be obtained explicitly as

C = GF.

Proof. Let us recall y(k) = z1(k + 1) − z1(k) = [A11 + A12C − I]z1(k) + [Ad11 + Ad12C]z1(k−τ(k))+D1ω(k). Consider the following Lyapunov-Krasovskii functional

V (k) = V1(k) + V2(k) + V3(k) + V4(k),

where

$$V_1(k) = z_1^T(k)F^T PF z_1(k),$$

$$V_2(k) = \sum_{s=k-\tau_m}^{k-1} \alpha^{s-k+1} z_1^T(s)F^T Q_1 F z_1(s) + \sum_{s=k-\tau_M}^{k-\tau_m-1} \alpha^{s-k+1} z_1^T(s)F^T Q_2 F z_1(s)$$

$$V_3(k) = \sum_{s=-\tau_m}^{-1} \sum_{v=k+s}^{k-1} \alpha^{v-k+1} \rho^T(v) R_1 \rho(v) + \sum_{s=-\tau_M}^{-\tau_m-1} \sum_{v=k+s}^{k-1} \alpha^{v-k+1} \rho^T(v) R_2 \rho(v),$$

$$V_4(k) = \sum_{s=-\tau_m}^{-1} \sum_{u=s}^{-1} \sum_{v=k+u}^{k-1} \alpha^{v-k+1} y^T(v)F^T S_1 F y(v) + \sum_{s=-\tau_M}^{-\tau_m-1} \sum_{u=s}^{-1} \sum_{v=k+u}^{k-1} \alpha^{v-k+1} y^T(v)F^T S_2 F y(v).$$

By taking the forward difference of V1(k) along the solutions of system, we have

$$\Delta V_1(k) = z_1^T(k+1)F^T PF z_1(k+1) - \alpha^{-1} z_1^T(k)F^T PF z_1(k) + (\alpha^{-1}-1)V_1$$

where z1(k + 1) = y(k) + z1(k). Therefore, ΔV1(k) can be obtained of the form

$$\Delta V_1(k) = \xi^T(k)\left[\Pi_1^T P \Pi_1 - \alpha^{-1} e_1^T P e_1\right]\xi(k) + (\alpha^{-1}-1)V_1$$

The forward differences of V2(k) and V3(k) along the solutions of system are obtained as

$$\Delta V_2(k) = z_1^T(k)F^T Q_1 F z_1(k) + \alpha^{-\tau_m} z_1^T(k-\tau_m)F^T(Q_2-Q_1)F z_1(k-\tau_m)$$
$$+\alpha^{-\tau_M} z_1^T(k-\tau_M)F^T Q_2 F z_1(k-\tau_M) + (\alpha^{-1}-1)V_2$$
$$= \xi^T(k)\left[e_1^T Q_1 e_1 + \alpha^{-\tau_m} e_2^T(Q_2-Q_1)e_2 + \alpha^{-\tau_M} e_4^T Q_2 e_4\right]\xi(k) + (\alpha^{-1}-1)V_2 \quad (5.14)$$

,

$$\Delta V_3(k) = \rho^T(k)R_c\rho(k) - \sum_{s=k-\tau_m}^{k-1} \alpha^{s-k}\rho^T(s)R_1 \rho(s) - \sum_{s=k-\tau_M}^{k-\tau_m-1} \alpha^{s-k}\rho^T(s)R_2\rho(s) + (\alpha^{-1}-1)V_3$$

and

.

$$\Delta V_3(k) = \rho^T(k)R_c\rho(k) - \sum_{s=k-\tau_m}^{k-1} \alpha^{s-k}\rho^T(s)R_1\rho(s) + (\alpha^{-1}-1)V_3$$
$$- \sum_{s=k-\tau_M}^{k-\tau(k)-1} \alpha^{s-k}\rho^T(s)R_2\rho(s) - \sum_{s=k-\tau(k)}^{k-\tau_m-1} \alpha^{s-k}\rho^T(s)R_2\rho(s)$$

For τm ≤ τ(k) ≤ τM,k ∈ Z+ and τm < τM,

we have

By using Lemma 1,the following estimation can be obtained as

$$-\sum_{s=k-\tau_m}^{k-1} \alpha^{s-k}\rho^T(s)R_1\rho(s) \le -\alpha_1 \begin{bmatrix} \sum_{s=k-\tau_m}^{k-1} F z_1(s) \\ F z_1(k) - F z_1(k-\tau_m) \end{bmatrix}^T R_1 \times \begin{bmatrix} \sum_{s=k-\tau_m}^{k-1} F z_1(s) \\ F z_1(k) - F z_1(k-\tau_m) \end{bmatrix}$$
$$\le -\alpha_1 \xi^T(k)\Pi_3^T R_1 \Pi_3 \xi(k).$$

Using the same argument, we also have

$$-\sum_{s=k-\tau(k)}^{k-\tau_m-1} \alpha^{s-k}\rho^T(s)R_2\rho(s) - \sum_{s=k-\tau_M}^{k-\tau(k)-1} \alpha^{s-k}\rho^T(s)R_2\rho(s)$$
$$\le -\frac{1-\alpha}{\alpha^{\tau_m+1}-\alpha^{\tau(k)+1}}\left[\sum_{s=k-\tau}^{k-\tau(k)-1}\rho(s)\right]^T R_2 \left[\sum_{s=k-\tau}^{k-\tau(k)-1}\rho(s)\right]$$
$$-\frac{1-\alpha}{\alpha^{\tau(k)+1}-\alpha^{\tau_M+1}}\left[\sum_{s=k-\tau(k)}^{k-\tau_m-1}\rho(s)\right]^T R_2 \left[\sum_{s=k-\tau(k)}^{k-\tau_m-1}\rho(s)\right]$$
$$\le -\frac{1-\alpha}{\alpha^{\tau_m+1}-\alpha^{\tau_M+1}}\left[\frac{1}{\beta_1}\eta_1^T R_2\eta_1 + \frac{1}{1-\beta_1}\eta_2^T R_2 \eta_2\right],$$

Where $\beta = \dfrac{\alpha^{\tau_m}-\alpha^{\tau(k)}}{\alpha^{\tau_m}-\alpha^{\tau_M}}$, and

$$\eta_1 = \begin{bmatrix} \sum_{s=k-\tau(k)}^{k-\tau_m-1} F z_1(s) \\ F z_1(k-\tau_m) - F z_1(k-\tau(k)) \end{bmatrix}, \eta_2 = \begin{bmatrix} \sum_{s=k-\tau_M}^{k-\tau(k)-1} F z_1(s) \\ F z_1(k-\tau(k)) - F z_1(k-\tau_M) \end{bmatrix}$$

.

Thus, from Lemma 2, we have the following estimation

$$2(z_1^T(k)F^T + \lambda y_1^T(k)F^T)\left[(A_{11}+A_{12}C-I)z_1(k) + (A_{d11}+A_{d12}C)z_1(k-\tau(k)) + D_1\omega(k) - y(k)\right] = 0$$

$$-\sum_{s=k-\tau(k)}^{k-\tau_m-1} \alpha^{s-k}\rho^T(s)R_2\rho(s) - \sum_{s=k-\tau_M}^{k-\tau(k)-1} \alpha^{s-k}\rho^T(s)R_2\rho(s)$$
$$\le -\alpha_2 \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}^T \begin{bmatrix} R_2 & X \\ \star & R_2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$
$$\le -\alpha_2 \xi^T(k)\Pi_4^T \begin{bmatrix} R_2 & X \\ \star & R_2 \end{bmatrix} \Pi_4 \xi(k).$$

Next, the difference of V4(k) along the solutions of system is calculated as follows.

$$\Delta V_4 = y^T(k)F^T(\tau_a S_1 + \tau_b S_2)Fy(k) + (\alpha^{-1}-1)V_4$$
$$- \sum_{s=-\tau_m}^{-1}\sum_{v=s}^{-1}\alpha^v y^T(k+v)F^T S_1 F y(k+v)$$
$$- \sum_{s=-\tau_M}^{-\tau_m-1}\sum_{v=s}^{-1}\alpha^v y^T(k+v)F^T S_2 F y(k+v).$$

By using Lemma 1, we have

$$- \sum_{s=-\tau_m}^{-1}\sum_{v=s}^{-1}\alpha^v y^T(k+v)F^T S_1 F y(k+v)$$
$$\leq -\alpha_3\left(\sum_{s=-\tau_m}^{-1}\sum_{v=s}^{-1} y(k+v)\right)^T F^T S_1 F\left(\sum_{s=-\tau_m}^{-1}\sum_{v=s}^{-1} y(k+v)\right)$$
$$\leq -\alpha_3\left(\tau_m z_1(k) - \sum_{s=k-\tau_m}^{k-1} z_1(s)\right)^T F^T S_1 F\left(\tau_m z_1(k) - \sum_{s=k-\tau_m}^{k-1} z_1(s)\right)$$
$$\leq -\alpha_3 \xi^T(k)\Pi_5^T S_1 \Pi_5 \xi(k).$$

Similarly, we also obtain

$$- \sum_{s=-\tau_M}^{-\tau_m-1}\sum_{v=s}^{-1}\alpha^v y^T(k+v)F^T S_2 F y(k+v)$$
$$\leq -\alpha_4\left[(\tau_M-\tau_m)z_1(k) - \sum_{s=k-\tau_M}^{k-\tau_m-1} z_1(s)\right]^T F^T S_2 F \times$$
$$\times\left[(\tau_M-\tau_m)z_1(k) - \sum_{s=k-\tau_M}^{k-\tau_m-1} z_1(s)\right]$$
$$\leq -\alpha_4\left[(\tau_M-\tau_m)z_1(k) - \sum_{s=k-\tau(k)}^{k-\tau_m-1} z_1(s) - \sum_{s=k-\tau_M}^{k-\tau(k)-1} z_1(s)\right]^T F^T S_2 F \times$$
$$\times\left[(\tau_M-\tau_m)z_1(k) - \sum_{s=k-\tau(k)}^{k-\tau_m-1} z_1(s) - \sum_{s=k-\tau_M}^{k-\tau(k)-1} z_1(s)\right]$$
$$\leq -\alpha_4 \xi^T(k)\Pi_6^T S_2 \Pi_6 \xi(k).$$

Moreover, for any scalar λ and a non-singular matrix F, by using the descriptor method, we always have the following equation

.

Note that from the above notations with some simple computations, equation can be rewritten in the form of

$$\xi(k)^T(\Gamma\mathcal{A}_c + \mathcal{A}_c^T\Gamma^T)\xi(k) = 0.$$

Finally, we obtain

ΔV (k) + (1 − α−1)V (k) − (1 − α−1)ωT (k)ω(k) ≤ ξT (k)Ω(α)ξ(k),∀k ∈ Z+.

Therefore, it follows from conditions and of Theorem that

ΔV (k) + (1 − α−1)V (k) − (1 − α−1)ωT (k)ω(k) ≤ 0

which yields

$$V(k) \leq \varpi_p^2 + V(0)e^{-\gamma k}\forall k \in \mathbb{Z}^+$$

.

From Lemma 3, we have

$$\lim_{k\to\infty} \sup V(k) \leq \varpi_p^2, \quad k \in \mathbb{Z}^+.$$

Thus, by using the spectral properties of symmetric positive-definite matrix, we obtain,

$$\lambda_{min}(F^T P F)\lim_{k\to\infty}\sup\|z_1(k)\|^2 \leq \lim_{k\to\infty}\sup V(k)$$
$$\leq \varpi_p^2.$$

This means that limk→∞ sup‖z1(k)‖ ≤ r. Thus, the induced sliding dynamics are bounded within a ball with radius r defined in . The proof is completed.

Remark 9 Note that the obtained conditions in Theorem 22 are also not LMIs and the solution to this problem can be found by using the method, presented in Remark 2.

Remark 10 As the radius of the ball B(0,r) in equation is $r = \frac{\varpi_p}{\sqrt{\delta}}$ determined by where δ = λmin(FT PF), to find the possible smallest radius r, one may proceed with a simple optimisation process as suggested in to maximise δ subject to δI ≤ FT PF, i.e., to formulate the following optimisation problem: minimise($\frac{\varpi_p}{\sqrt{\delta}}$) subject to

$$\begin{cases} (a) \quad F^T PF \geq \delta I \\ (b) \quad (5.9a) \end{cases}$$ and

Note that inequality FT PF ≥ δI is equivalent to (K−1)T P(K−1) ≥ δI. Pre- and post- multiplying this inequality by KT and its transpose, respectively, we obtain

−P + δKT K ≤ 0.

By using the Schur complement, we have

$$\begin{bmatrix} -P & K^T \\ \star & -\frac{1}{\delta}I \end{bmatrix} \leq 0$$.

IV. ROBUST QUASI-SLIDING MODE CONTROLLER DESIGN

In discrete-time quasi-sliding mode control, under the appropriate controller, the system trajectory, starting from any initial state, will be driven towards the sliding surface in finite time. After reaching the sliding surface, the state trajectories cross the sliding surface for the first time, and repeat that again in successive sampling periods, resulting in a zigzag motion along the sliding surface. This motion will be bounded inside a specified region, the so-called quasi-sliding mode band (QSMB)[12]. In the previous section, under appropriate conditions, in the sliding mode, the state trajectories of the system are convergent within a ball whose radius can be minimised. In the following, the objective is to design a robust discrete-time quasi-sliding mode controller to drive the system dynamics towards the above ball in finite time and maintain it there afterwards.

First, it is noted from that the external disturbance ω(k) is bounded, and so is the uncertain term d(k) = C Dω(k). Without loss of generality, we have, componentwise:

dm ≤ d(k) ≤ dM.

From a physical perspective, by assuming the boundedness of z(k−τ(k)), and hence

of vector a(k) = C Adz(k − τ(k)). We then have, similarly:

am ≤ a(k) ≤ aM.

define

$$a_0 = \frac{a_m + a_M}{2}, \; d_0 = \frac{d_m + d_M}{2} \qquad 5.28)$$

$$a_1 = \frac{a_M - a_m}{2}, \; d_1 = \frac{d_M - d_m}{2}.$$

From sliding function , in which the design matrix C = [C1 C2 ...Cm]T is obtained from (5.11), we have s(k) = [s1(k) s2(k) ... sm(k)]T , where si(k) = −Ciz1(k) + z2i(k) and Ci is a row vector in R1×(n−m).

Theorem 23 For given positive integers τm and τM (5.24) of the delay, satisfying 0 < τm < τM, and a bound ωp > 0 of the external disturbance, if there exist scalars λ and α, where α > 1, a feasible solution of X,K,G,P and Qj,Rj,Sj, j = 1,2, for matrix inequalities , and , and with the sliding function chosen as in for sliding motion , the state trajectories of system are driven towards the sliding surface in a finite time under the following control law:

$$u(k) = -(\overline{C}\overline{B})^{-1}\left[\overline{C}\overline{A}z(k) - (I - qT_s)s(k) + a_0 + d_0 + (\varepsilon T_s + a_1 + d_1)\circ sgn(s(k))\right] \qquad (5.29)$$

,

where sgn(s(k)) = [sgn(s1(k)),sgn(s2(k)),...,sgn(sm(k))]T , Ts is the sampling period, q = diag(q1, q2,..., qm) and ε = [ε1,ε2,...,εm]T , in which positive scalars qi and εi, i = 1,2,...,m, are chosen such that 1−Tsqi > 0 for quasi-sliding mode bands Δi(k) given by

$$\Delta_i = \frac{\varepsilon_i T_s}{1 - T_s q_i}.$$

Proof. From the designed sliding function s(k) = Cz(k), we have

$$\Delta s(k) = \overline{C}z(k+1) - \overline{C}z(k)$$

$$= \overline{C}\left[(\overline{A} - I)z(k) + \overline{A}_d z(k - \tau(k)) + \overline{B}u(k) + \overline{D}\omega(k)\right]$$

.

By substituting the control law into equation , we obtain

Δs(k) = − qTss(k) − εTs ◦ sgn(s(k))

+ [a(k) − a0 − a1 ◦ sgn(s(k))]

+[d(k) − d0 − d1 ◦ sgn(s(k))].

Now, we have to show that the proposed control scheme satisfies the reaching condition and the existence of the quasi-sliding mode is guaranteed. This requires that the sign of the incremental change Δs(k) = s(k + 1) − s(k) should be opposite to the sign of s(k), componentwise.
It is easy to see that when s(k) > 0, we have

$$a(k) \le a_0 + a_1,$$

$$d(k) \le d_0 + d_1,$$

and when $s(k) < 0$,

$$a(k) \ge a_0 - a_1,$$

$$d(k) \ge d_0 - d_1.$$

Thus, by judging the sign of the four terms constituting Δs(k) in , we can see that the sign of the increment Δs(k) of is always opposite to the sign of s(k), componentwise. Thus, if design parameters qi > 0 and εi > 0 are chosen with 1 − qiTs > 0, i = 1,2,...,m, then a quasi-sliding mode exists with quasi-sliding

mode bands $\Delta_i = \frac{\varepsilon_i T_s}{1-q_i T_s}$ [36]. This completes the proof.

Remark 11 It is worth mentioning that in this chapter, the quasi-sliding mode control law is obtained from the sliding function , whereby the design matrix C can be computed directly from after the solution of conditions and the optimisation process mentioned in Remark 2. This gives designers a certain liberty in selecting the controller parameters in for a desired QSMB as compared to existing methods in the literature, where the QSMB is determined subsequently from the design of a quasi-sliding mode control law.

Example

Consider a truck-trailer system for the case of unmatched external disturbance
which was given in as follows

$$x(k+1) = \begin{bmatrix} 1.3461 & 0 & 0 \\ 0.3461 & 1 & 0 \\ 0.0086 & -0.05 & 1 \end{bmatrix} x(k) + \begin{bmatrix} -0.0384 & 0 & 0 \\ 0.0384 & 0 & 0 \\ 0.0001 & 0 & 0 \end{bmatrix} x(k-\tau(k))$$
$$+ \begin{bmatrix} -0.5747 \\ 0 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ -0.01 \\ 0.01 \end{bmatrix} \omega(k),$$

where x(k) = [x1(k) x2(k) x3(k)]T is the system state vector of the angle difference between the trailer and the truck, the angle of the trailer, and the vertical position of the rear end of the trailer, respectively. The control input signal u(k) is the steering angle. Here, the truck-trailer system is assumed to be subject to an external disturbance ω(k) with an upper bound ωp = 0.3. The control objective is to minimise the effects of time-varying delay and unmatched disturbances, while backing the trailertruck along the horizontal line x3(k) = 0 in a safe and robust manner. Note that the proposed approaches in are available for linear discrete-time systems with time-varying delay and matched disturbances. Therefore, it can not apply for this case. For this, the sampling period is chosen as Ts = 0.1 sec. Matrix T of the transformation z(k) = Tx(k) can be obtained from a singular value decomposition
of matrix B as:

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

From Theorem 22 and Remarks 9 and 10, by choosing α = 1.15,λ = 0.6 and solving matrix inequalities and , we obtain the following matrices

$$G = 1.0e-003 * [-0.6807 \ \ 0.0009], K = 1.0e-003 * \begin{bmatrix} 0.9842 & 0.2487 \\ 0.2701 & 0.1449 \end{bmatrix}$$

.

Thus, the switching gain is calculated as C = [−1.3105 2.2547]. As a result, the sliding function is obtained as

s(k) = [−1.3105 2.2547 1]z(k).

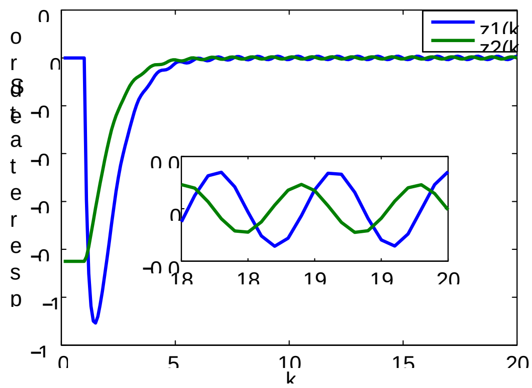
Moreover, the possible smallest radius of the ball which bounds the state trajectories of the reduced-order system can be obtained as r = 0.01. With the assumption of boundedness of the external disturbance ωp = 0.3, the average and variation magnitude of the disturbance-related uncertainty d(k) can be found as

d0 = 0,d1 = 0.0107.

Similarly, we have for the delay-related uncertain term a(k) a0 = 0 and a1 = 0.0012. By choosing q = 2 and ε = 0.016 for a QSMB of Δ = 0.002, from Theorem 23, the robust discrete-time quasi-sliding mode controller is obtained in the form

$$u(k) = 1.74\left(\left[1.4232\ \ -2.2547\ \ 1.8190\right]z(k) - 0.8s(k) + 0.0119\,\mathrm{sgn}(s(k))\right). \quad (5.33)$$

With an initial condition of the system of x(k) = [0.2 0 − 0.85]T , and the timevarying delay τ(k) is a random integer belonging to the interval , the state responses of the reduced-order system via z(k) and closed-loop system are shown in figures and respectively. It can be seen in the inset of that after reaching the sliding surface, the states trajectories of the reduced-order system exponentially converges within a ball with radius r ≤ 0.01 in spite of time-varying delay and unmatched external disturbances. The responses of the control input signal and the sliding surface are depicted respectively to illustrate the steering process of the truck-trailer system. These indicate that



State responses of the reduced-order system with unmatched disturbances the effects of time-varying delay and unmatched bounded disturbances have been successfully suppressed by using the proposed controller.

Example

Now, consider the truck-trailer system in the case of without external disturbances (i.e., ω(k) = 0). The control objective is still the same. From Theorem 22, by choosing α = 1.15,λ = 1.4 and solving matrix inequalities, we obtain the following matrices

$$G = 1.0e-003*\left[-0.3920\ \ -0.0152\right], K = 1.0e-003*\begin{bmatrix} 0.8726 & 0.3033 \\ 0.3193 & 0.2134 \end{bmatrix}$$

.

Thus, the design matrix C is calculated as C = [−0.8817 1.1819]. As a result, the sliding surface is obtained as

s(k) = [−0.8817 1.1819 1]z(k).

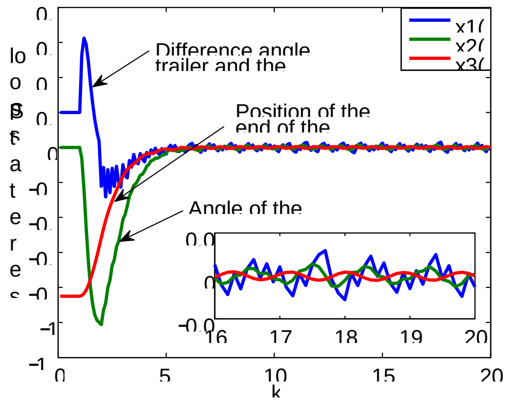


Figure: State responses of the closed-loop system with unmatched disturbance

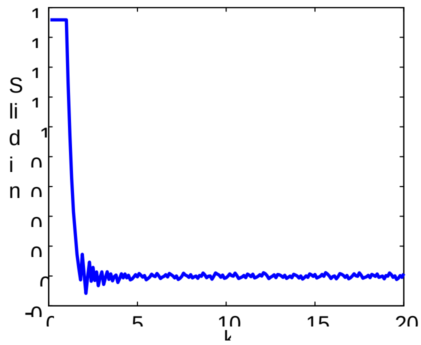
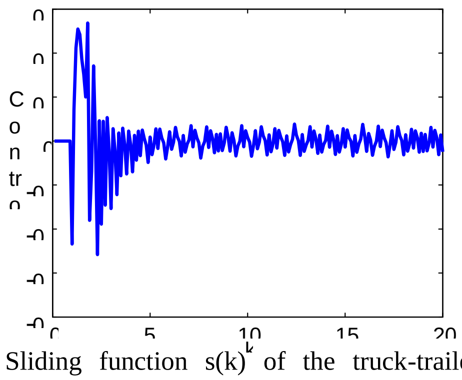


Figure : Steer angle u(k) of the truck-trailer system with unmatched disturbances



Sliding function s(k) of the truck-trailer system with unmatched disturbances Similarly, the bound of uncertainty a(k) is determined as a1 = 7.9694x10−5. By using the same controller parameters as in. Theorem , the robust discrete-time quasi-sliding mode controller is obtained of the form

$$u(k) = 1.74\left(\begin{bmatrix}0.9408 & -1.1819 & 1.6614\end{bmatrix} z(k) - 0.8s(k) + 7.9694e-005\,\mathrm{sgn}(s(k))\right). \tag{5.34}$$

For the sake of demonstration the effectiveness of the proposed control schemes in terms of robustness to time-varying delay, the initial conditions x(k) = [0.1 0 −0.1]T will be used. The obtained conditions are still feasible with an interval time-varying delay [τm τM], where τM ≤ 16. The state responses of the closed-loop system exponentially converge to the origin. The control input signal and sliding surface are depicted in figures. It can be seen clearly that the close-loop control system is robustly stable with a large range of time-varying delay. It is worth pointing out that by comparing simulation results with Example The proposed control scheme is more effective for the truck-trailer

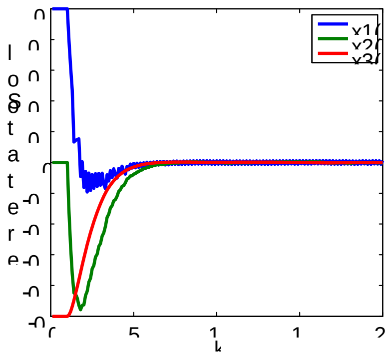


Figure : State responses of the closed-loop system without external disturbances system in the absence of external disturbances.

V. Conclusion

In this chapter, the problem of robust discrete-time quasi-sliding mode control design for a class of linear discrete-time systems with time-varying delay and unmatched disturbances has been addressed. Based on the Lyapunov-Krasovskii method, combined with the reciprocally convex approach, sufficient conditions for the existence of a stable sliding surface are derived in terms of matrix inequalities. These conditions also guarantee that the effects of time-varying delay and unmatched disturbances are mitigated when the system is in the sliding mode. Finally, a discrete-time quasi-sliding mode controller is proposed to satisfy the reaching condition. Numerical examples are provided to illustrate the feasibility of the proposed approach.

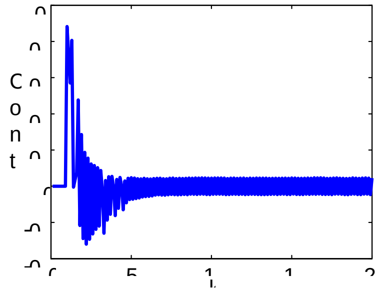


Figure **:** Steer angle u(k) of the truck-trailer system without external disturbances

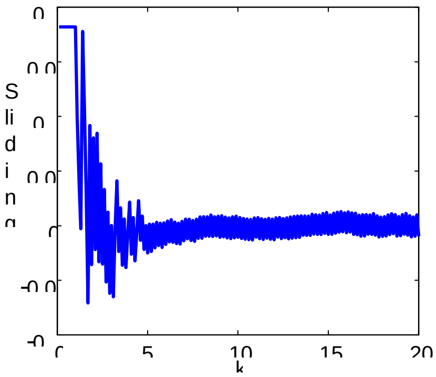


Figure : Sliding function s(k) of the truck-trailer system without external disturbances

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