Generation of Anti-Fractals in SP-Orbit

Mandeep Kumari[1], Sudesh Kumari[2], Renu Chugh[3]

[1,2,3]Department of Mathematics, Maharshi Dayanand University, Rohtak-12400, Haryana, India

Abstract — *In this paper we generate a new class of Tricorns and Multicorns using SP iteration (a four-step feedback process) and explore the geometry of superior antifractals. Other researchers have already generated antifractals using Picard, Mann, ishikawa and Noor orbits that are examples of one – step, two-step, three-step and four-step feedback processes.*

Keywords — *Antipolynomial, antifractal, Tricorn, Multicorn, SP-orbit.*

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I. Introduction

Fractals are defined as "objects that appear to be broken into a number of pieces and each piece is a copy of the entire shape". "Fractal" is the word taken from the Latin word "fractus" which means "broken". The term "fractal" was first used by a young mathematician, Mandelbrot [2]. Julia introduced the concept of iterative function and he derived the Julia set in 1919. After that, in 1979, Mandelbrot [2] extended the work of Gaston Julia and introduced the Mandelbrot set, a set of all connected Julia sets. Many researchers have studied Julia sets and Mandelbrot sets from different aspects.

The connected locus of antipolynomial $z \to \bar{z}^m + c$ is known as Tricorn. The term Tricorn was firstly used by Milnor. In 2003, Shizuo et al.[15] described various properties of Tricorn and Multicorn by computing beautiful figures and quoted that Multicorns are the generalized Tricorns or the Tricorns of higher order.

The dynamics of antiholomorphic complex polynomials $z \to \bar{z}^m + c$, for $m \geq 2$, was studied and explored to visualize interesting Tricorns and Multicorns antifractals with respect to one-step feedback process [12], two step-feedback process [10, 11], three-step feedback process [18] and four step feedback process[1,3].

The dynamics of antipolynomial $z \to \bar{z}^m + c$ where $m \geq 2$ with respect to iterative function generates amazing Tricorn and Multicorns [12, 14, 15]. Crowe et. al. [16] considered it as a formal analogy with Mandelbrot sets and named it as Mandelbar set. They also brought their bifurcation features along arcs rather than at points. Multicorns have been found in a real slice of the cubic connectedness locus [15]. Winter [13] showed that the boundary of the Tricorn contains arc. The symmetries of Tricorn and Multicorns have been analyzed by Lau and Schleicher [4].

In 2011, W. Phuengrattana and S. Suantai [18] proposed the SP-iteration for approximating a fixed point of continuous functions on an arbitrary interval. They compared the convergence speed of Mann, Ishikawa, Noor and SP-iterations using some numerical examples and proved that the SP-iteration is equivalent to and converges faster than the other iterations. In this paper we generate a new class of Tricorns and Multicorns under SP orbit which is an example of four-step feedback process and analyze them.

II. Preliminaries

***Definition* 1**. [12] (Multicorn). The multicorns A_c for the quadratic function $A_c(z) = \bar{z}^m + c$ is defined as the collection of all $c \in C$ for which the orbit of the point 0 is bounded, that is

$$A_c = \left\{ c \in C : A_c^n(0) \text{ does not tend to } \infty \right\}$$

where C is a complex space. A_c^n is the n^{th} iterate of the function $A_c(z)$. An equivalent formulation is that the connectedness of loci for higher degree antiholomorphic polynomials $A_c(z) = \bar{z}^m + c$ are called multicorns.

Note that at m = 2, multicorns reduce to tricorn. Naturally, the tricorns lives in the real slice $d = \bar{c}$ in the two dimensional parameter space of maps $z \to \left(z^2 + d\right)^2 + c$.

They have (m+1)-fold rotational symmetries. Also, by dividing these symmetries, the resulting multicorns are called unicorns [14].

***Definition* 2.** [6] (Julia Set). The filled in Julia set of the function g is defined as

$$K(g) = \{z \in C: g^k(z) \text{ does not tend to } \infty\},$$

where C is the complex space, $g^k(z)$ is k^{th} iterate of function g and $K(g)$ denotes the filled Julia set. The Julia set of the function g is defined to be the boundary of $K(g)$, i.e.,

$$J(g) = \partial K(g),$$

where $J(g)$ denotes the Julia set.

***Definition* 3**. [12] (Mandelbrot Set). The Mandelbrot set M consists of all parameters c for which the filled Julia set of $Q_c(z) = z^2 + c$ is connected, that is

$$M = \{c \in C: K(Q_c) \text{ is connected }\}.$$

In fact, M contains an enormous amount of information about the structure of Julia sets. The Mandelbrot set M for the Quadratic $Q_c(z) = z^2 + c$ is defined as the collection of all $c \in C$ for which the orbit of the point 0 is bounded, that is

$$M = \{c \in C: \{Q_c^n(0)\}\ ; n = 0, 1, 2, \ldots \text{ is bounded}\}.$$

We choose the initial point 0 as 0 is the only critical point of Q_c.

Now, we give definition of the SP orbit, which will be used in the paper to implement four-step feedback process in the dynamics of polynomial $z \to \bar{z}^m + c$.

Definition 4. [8] Let $T : X \to X$ be a mapping. Let us consider a sequence $\{z_n\}$ of iterates for initial point $z_0 \in X$ such that

$\{z_{n+1}: \; z_{n+1} = (1 - \alpha_n)\, u_n + \alpha_n T u_n\,;$
$u_n = (1- \beta_n)\, v_n + \beta_n T v_n\,;$
$v_n = (1-\gamma_n)\, z_n + \gamma_n T z_n\,; n = 0, 1, 2, \ldots\},$

where α_n, β_n, $\gamma_n \in [0, 1]$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences of positive numbers. The above sequence of iterates is called as SP orbit, which is a function of five tuples $(T, z_0, \alpha_n, \beta_n, \gamma_n)$.

III. MAIN RESULT

Now, we will obtain a general escape criterion for polynomials of the form $G_c(z) = z^m + c$

Theorem 1. For a general function $G_c(z) = z^m + c$, m =1, 2, 3…, where $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \gamma < 1$, and c is a complex number. Define

$$z_1 = (1-\alpha)u + \alpha G_c(u)$$

$$z_2 = (1-\alpha)u_1 + \alpha G_c(u_1)$$

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$$z_m = (1-\alpha)u_{m-1} + \alpha G_c(u_{m-1}),$$

where $m = 1, 2, 3, 4, \ldots$

Then, the general Superior escape criterion is $\max\{|c|, (2/\alpha)^{1/m-1}, (2/\beta)^{1/m-1}, (2/\gamma)^{1/m-1}\}$.

Proof. For proving the theorem, we shall use the method of induction.

For m = 1, we have $G_c(z) = z + c$, and this implies $|z| > \max\{|c|, 0, 0, 0\}$.

For m = 2, we have $G_c(z) = z^2 + c$, then the escape criterion is

$$|z| > \max\{|c|, 2/\alpha, 2/\beta, 2/\gamma\}.$$

Similarly, for m = 3, we get $G_c(z) = z^3 + c$. The escape criterion is

$$|z| > \max\{|c|, (2/\alpha)^{1/2}, (2/\beta)^{1/2}, (2/\gamma)^{1/2}\}.$$

Hence the theorem is true for m=1, 2, 3 …

Now, suppose that theorem is true for any m. We prove that the result is true for m+1.

Let $G_c(z) = z^{m+1} + c$ and $|z| \ge |c| > (2/\alpha)^{1/m}$, $|z| \ge |c| > (2/\beta)^{1/m}$ and $|z| \ge |c| > (2/\gamma)^{1/m}$.

Then, consider

$$|v| = |(1-\gamma)z + \gamma G_c(z)|,$$

where $G_c(z) = z^{m+1} + c$

$$= \left|(1-\gamma)z + \gamma(z^{m+1} + c)\right|$$

$$= |z|\left|\gamma\left|z^m\right| - \gamma + 1\right| - \gamma|c|$$

$$= |z|\left(\gamma\left|z^m\right| - 1\right) + \gamma|z| - \gamma|z| \quad (\because\ |z| \ge |c|)$$

i.e. $$|v| = |z|\left(\gamma\left|z^m\right| - 1\right) \tag{1}$$

Also, $|u| = |(1-\beta)v + \beta G_c(v)|$

$$= \left|(1-\beta)v + \beta\left(v^{m+1} + c\right)\right|$$

$$\ge \left|(1-\beta)|z|(\gamma|z|^m - 1) + \beta\left[\left\{|z|\left(\gamma|z|^m - 1\right)\right\}^{m+1} + c\right]\right| \quad \text{....(2)}$$

Since

$$|z| \ge (2/\gamma)^{1/m} \textit{ implies } \gamma|z|^m - 1 > 1, \tag{3}$$
$$\textit{so } |z|\left(\gamma|z|^m - 1\right) > |z|.$$

Using (3) in (2), we have

$$|u| \ge \left|(1-\beta)|z| + \beta\left(|z|^{m+1} + c\right)\right|$$

$$\ge \left|\beta|z|^{m+1} + (1-\beta)|z|\right| - |\beta c|$$

$$\ge \left|\beta|z|^{m+1} + (1-\beta)|z|\right| - \beta|z| \quad (\because |z| \ge |c|)$$

$$\ge |z|\left(\beta|z|^m - 1\right)$$

i.e. $$|u| \ge |z|\left(\beta|z|^m - 1\right) \tag{4}$$

Now for $z_m = (1-\alpha)u_{m-1} + \alpha G_c(u_{m-1})$, we have

$$|z_1| = |(1-\alpha)u + \alpha G_c(u)|$$

$$= \left|(1-\alpha)u + \alpha\left(u^{m+1} + c\right)\right|$$

$$\ge \left|(1-\alpha)|z|(\beta|z|^m - 1) + \alpha\left[\left\{|z|\left(\beta|z|^m - 1\right)\right\}^{m+1} + c\right]\right|$$

(5)

Since $|z| \geq (2/\beta)^{1/m}$ *implies* $\beta|z|^m - 1 > 1,$ *so* $|z|\left(\beta|z|^m - 1\right) > |z|.$ (6)

Using (6) in (5), we get

$$|z_1| \geq \left|(1-\alpha)|z| + \alpha\left(|z|^{m+1} - 1\right)\right|,$$

$$\geq \left|\alpha|z|^{m+1} + (1-\alpha)|z|\right| - \alpha|z|$$

$$\geq |z|\left(\alpha|z|^m - 1\right)$$

i.e., $|z_1| \geq |z|\left(\alpha|z|^m - 1\right)$

Since $|z| > (2/\alpha)^{1/m}$, $|z| > (2/\beta)^{1/m}$ and $|z| > (2/\gamma)^{1/m}$ exist, we have $\alpha|z|^m - 1 > 1 + \lambda > 1.$

In particular, $|z_1| > (1+\lambda)|z|$

$$\vdots$$

$$|z_m| > (1+\lambda)^m |z|$$

Hence,

$$|z_m| \to \infty \ as \ m \to \infty.$$

This completes the proof.

Corollary 1.1. Suppose $|c| > (2/\alpha)^{1/m-1}$, $|c| > (2/\beta)^{1/m-1}$ and $|c| > (2/\gamma)^{1/m-1}$ exists. Then the orbit $SP(G_c, 0, \alpha, \beta, \gamma)$ escapes to infinity.

Corollary 1.2. (Escape Criterion). Let us Assume that for some $k \geq 0$,

$|z_k| > \max\{|c|, (2/\alpha)^{1/k-1}, (2/\beta)^{1/k-1}, (2/\gamma)^{1/k-1}\}$, then $|z_k| > \lambda|z_{k-1}|$ and $|z_m| \to \infty$ as $m \to \infty$.

This corollary gives an algorithm to generate antiJulia sets for the functions of the type $G_c(z) = \bar{z}^m + c$, m = 2, 3, …

Thus, for visualizing new antifractals, the required escape criterion with respect to the SP orbit for $z \to \bar{z}^m + c$ is

$$\max\left\{|c|, (2/\alpha)^{\frac{1}{m-1}}, (2/\beta)^{\frac{1}{m-1}}, (2/\gamma)^{\frac{1}{m-1}}\right\} [7].$$

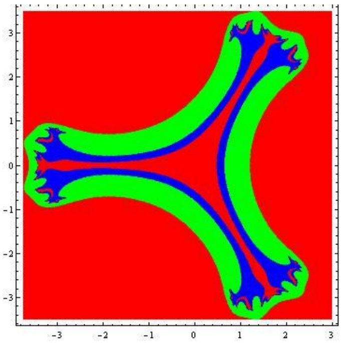
IV. MULTICORNS IN SP ORBIT

All In this section, we generate Tricorns and Multicorns by programming the polynomial $z \to \bar{z}^m + c$ in the software Mathematica 9.0 under SP orbit (see Figs. 1-18).

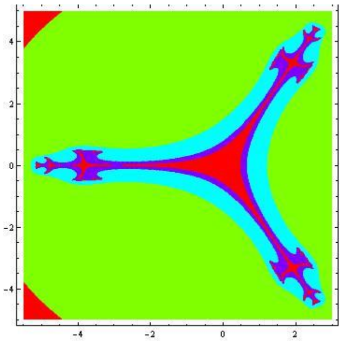
We have the following observations:

- The number of branches in the tricorns and multicorns is m+1, where m is the power of z. Also, few branches have m sub-branches.
- The shapes of Tricorns and Multicorns become different as we change the values of parameters.
- We have the beautiful Rangoli Patterns (Figs. 16, 17).
- We also find that higher degree multicorns become circular saw (Fig. 18).

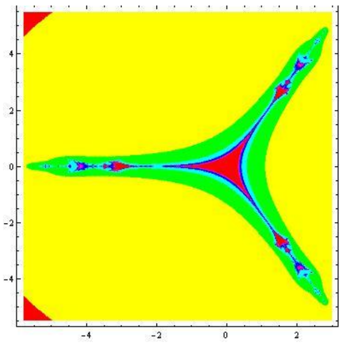
Some authors [1,3,11] had also found the similar conclusion while generating Multicorns using two-step, three-step, four-step feedback processes. The name circular saw was, first, given by Rani and Kumar to Mandelbrot sets [9].



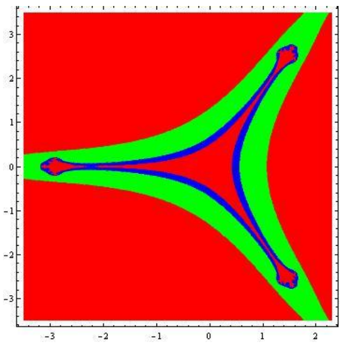
$Fig.1$: $\alpha = 0.3, \beta = 0.5, \ \gamma = 0.6$



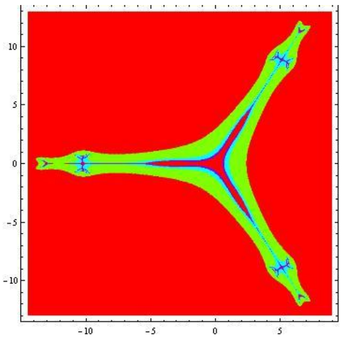
$Fig.2$: $\alpha = 0.6, \ \beta = 0.3, \ \gamma = 0.5$



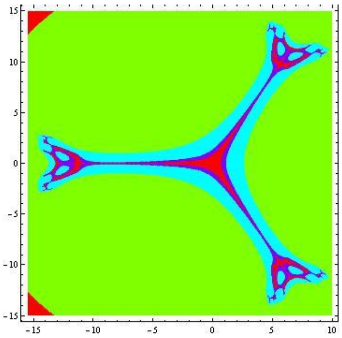
$Fig.3$: $\alpha = 0.68, \beta = 0.27, \gamma = 0.95$



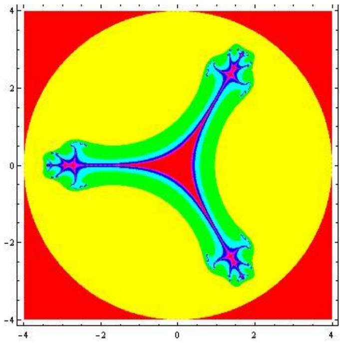
$Fig.4$: $\alpha = 0.95, \beta = 0.27, \ \gamma = 0.68$



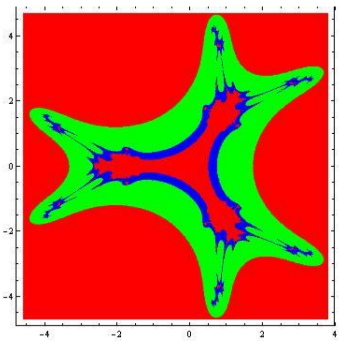
$Fig.5: \ \alpha = 0.9, \beta = 0.1, \gamma = 0.1$



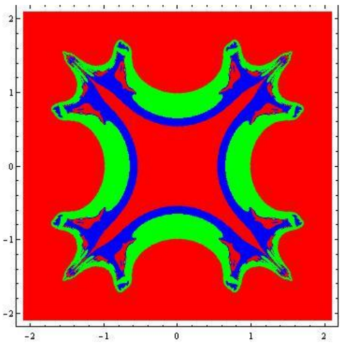
$Fig.6: \ \alpha = 0.1, \beta = 0.1, \gamma = 0.9$



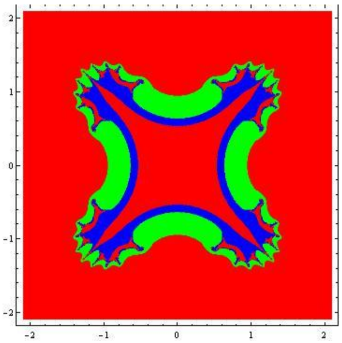
$Fig.7: \ \alpha = 0.5, \beta = 0.5, \gamma = 0.5$



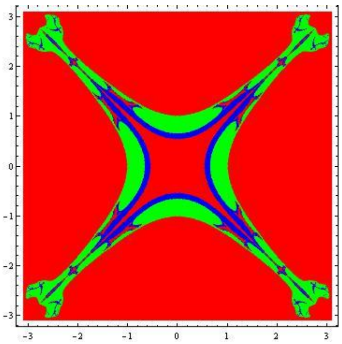
$Fig.8: \ \alpha = 0.09, \beta = 0.7, \gamma = 0.8$



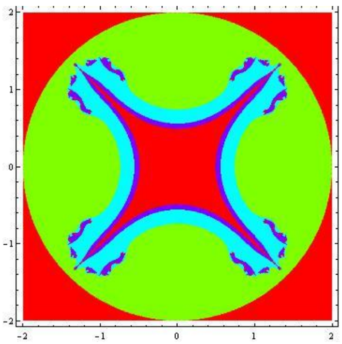
$Fig.9: \ m = 3, \ \alpha = 0.08, \ \beta = \gamma = 0.6$



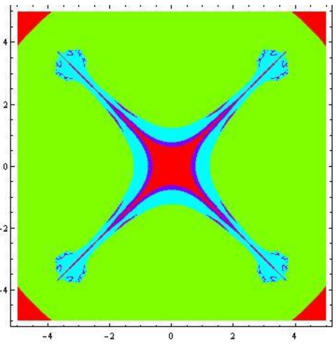
$Fig.10: \ m = 3, \ \alpha = \beta = 0.6, \gamma = 0.08$



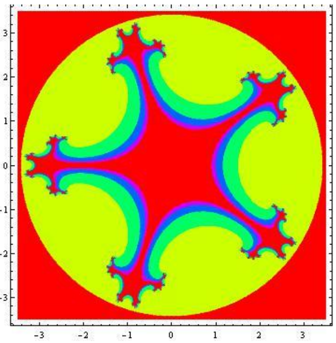
$Fig.11: \ m = 3, \ \alpha = 0.6, \beta = 0.08, \gamma = 0.6$



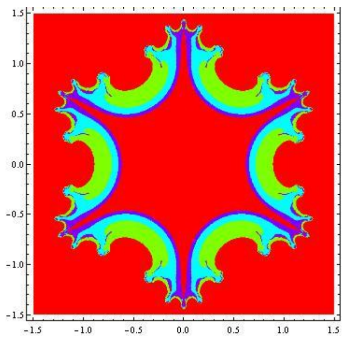
$Fig.12: \ m = 3, \ \alpha = 0.5, \beta = 0.5, \gamma = 0.5$



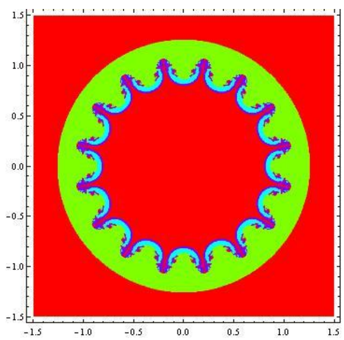
$Fig.13: m=3, \alpha=0.3, \beta=0.05, \gamma=0.6$



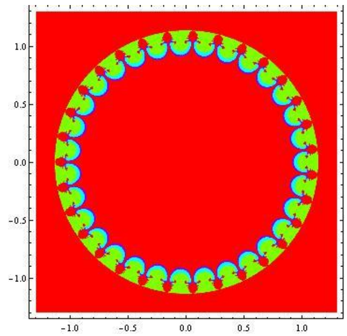
$Fig.14: m=4, \alpha=0.05, \beta=0.05, \gamma=0.05$



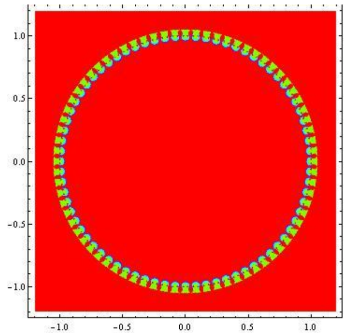
$Fig.15: m=5, \alpha=0.05, \beta=0.6, \gamma=0.3$



$Fig.16: m=15, \alpha=0.6, \beta=0.6, \gamma=0.08$



$Fig.17: m=30, \alpha=\beta=0.05, \gamma=0.2$

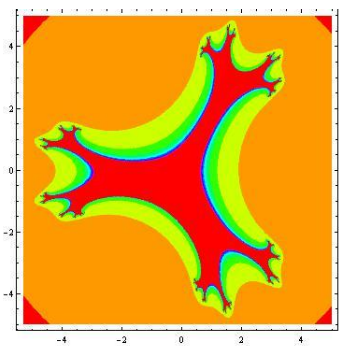


$Fig.18: Circular\ saw\ multicorn\ for$
$m=75, \alpha=\beta=\gamma=0.05$

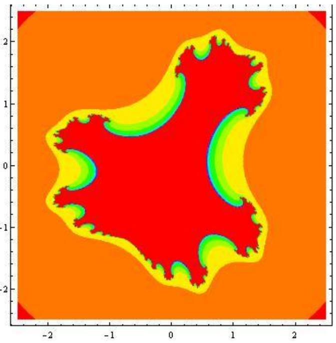
V. New Anti Julia Sets

We compute anti Julia sets for $z \to \overline{z}^m + c$ in the software Mathematica 9.0 via SP orbit. We have the following observations while generating them.

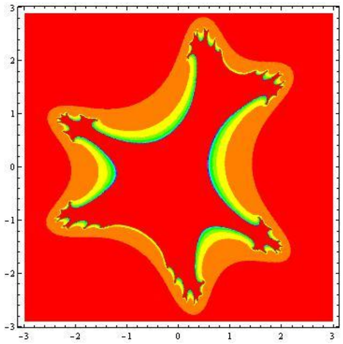
- In Figs.19-20, we notice that as we increase the value of parameters α, β and γ keeping constant c same anti Julia sets become fattier.
- The number of branches in anti Julia sets is *m*+1, where *m* is the power of *z*. Also, few branches have *m* sub-branches (see Figs. 27, 28, 29).
- Also, we observe that the higher degree anti Julia sets take different shapes (like circular shaw and Rangoli pattern) for different values of m, α, β, γ *and c.* (see Figs. 30-32)



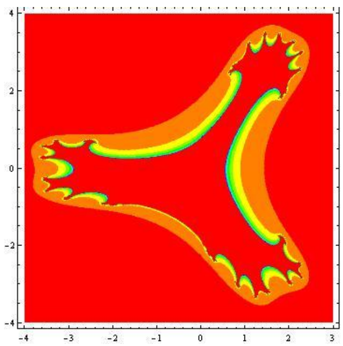
$Fig.19: AntiJulia\ set\ for\ m=2$
$\alpha=\beta=\gamma=0.3, c=0.3+0.5I$



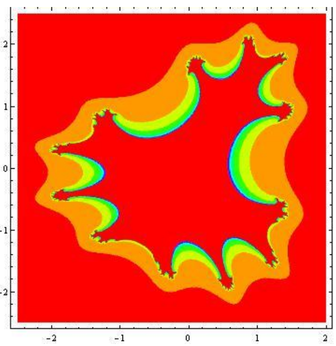
$Fig.\,20: AntiJulia\ set\ \ for\ m=2$
$\alpha=\beta=\gamma=0.6, c=0.3+0.5I$



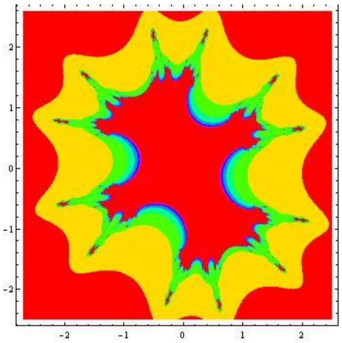
$Fig.\,21: AntiJulia\ set\ \ for\ m=2$
$\alpha=0.3, \beta=0.6, \gamma=0.9, c=0.3+0.5I$



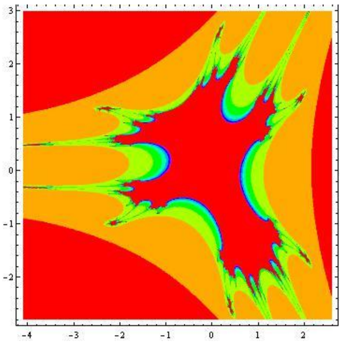
$Fig.\,22: AntiJulia\ set\ \ for\ m=2$
$\alpha=0.6, \beta=0.3, \gamma=0.9, c=0.3+0.5I$



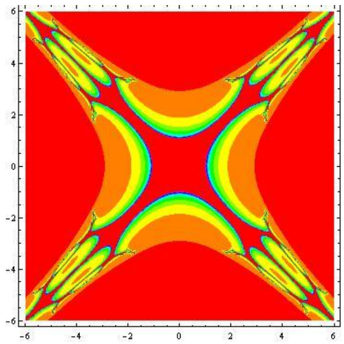
$Fig.\,23: AntiJulia\ set\ \ for\ m=2$
$\alpha=0.9, \beta=0.6, \gamma=0.3, c=0.3+0.5I$



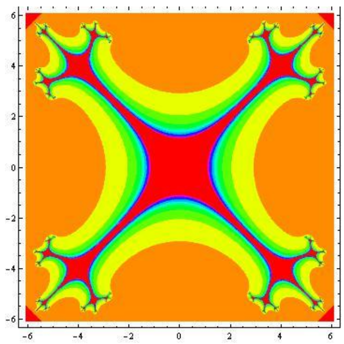
$Fig.\,24: AntiJulia\ set\ \ for\ m=2$
$\alpha=0.5, \beta=0.9, \gamma=0.1, c=0.3-0.5I$



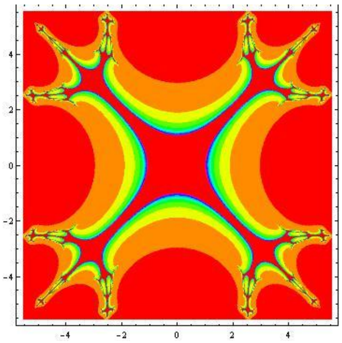
$Fig.\,25: AntiJulia\ set\ \ for\ m=2$
$\alpha=0.9, \beta=0.1, \gamma=0.5, c=0.3-0.5I$



$Fig.\,26: AntiJulia\ set\ \ for\ m=3$
$\alpha=0.9, \beta=0.1, \gamma=0.5, c=0.1-0.1I$

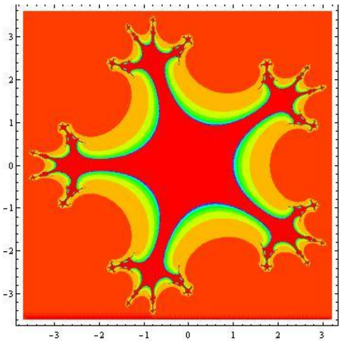


$Fig.\,27: AntiJulia\ set\ \ for\ m=3$
$\alpha=\beta=\gamma=0.03, c=0.1-0.1I$



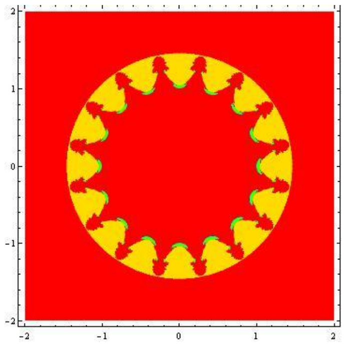
$Fig.28: AntiJulia\ set\ \ for\ m=3$

$\alpha=0.1, \beta=0.5, \gamma=0.9, c=0.1-0.1I$



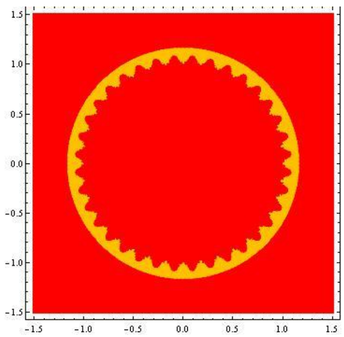
$Fig.29: AntiJulia\ set\ \ for\ m=4$

$\alpha=0.1, \beta=0.5, \gamma=0.9, c=0.1-0.1I$



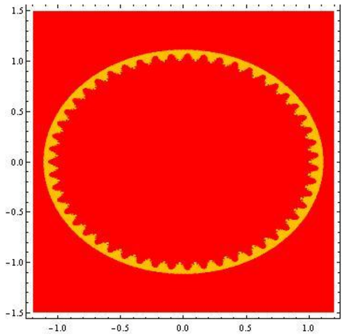
$Fig.30: AntiJulia\ set\ \ for\ m=15$

$\alpha=0.9, \beta=0.1, \gamma=0.5, c=0.1-0.1I$



$Fig.31: AntiJulia\ set\ \ for\ m=35$

$\alpha=0.5, \beta=0.9, \gamma=0.1, c=0.1-0.1I$



$Fig.32: Circular\ saw\ AntiJulia\ set\ \ for$

$m=50, \alpha=0.1, \beta=0.5, \gamma=0.9, c=0.05+0.05I$

VI. Conclusions

In the dynamics of antipolynomials $z \to \overline{z}^m + c$, where m > 2, there exist many antifractals for the same value of m but different values of parameters in SP orbit. In our results, we find that for higher degree polynomials, all the antifractals become circular saw. We observe that Multicorns are symmetrical about both x and y axis for odd values of m, but for even values of m, the symmetry is maintained only along x-axis.

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